Aspects of Canonical Formalism

- → Poisson brackets – a Classical perspective
- → Intro to Canonical Transformations
- → Intro to Action-Angle Variables and Integrability

Poisson Brackets and Canonical Transformations

a.) Poisson Brackets

- Fundamental notion of Hamiltonian Mechanics

i.e.:

- phase volume conservation
- incompressibility of phase space flow

→ "Liouville's Thm."

i.e. $\underline{V}_{\Gamma} = (\dot{q}_i, \dot{p}_i)$

$$\underline{\nabla}_{\Gamma} \cdot \underline{V}_{\Gamma} = \frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$$

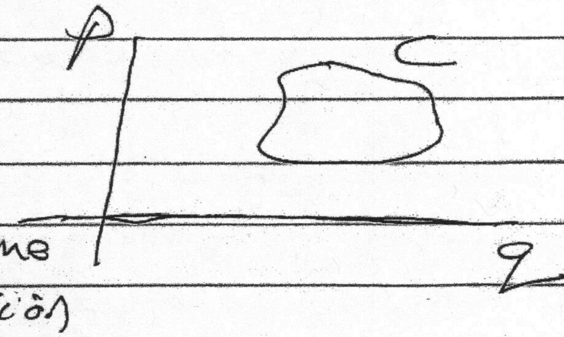
(1D)

equivalent to:

$$\int_C dp_i\, dq_i = \text{const.}$$

C ↓ area within C

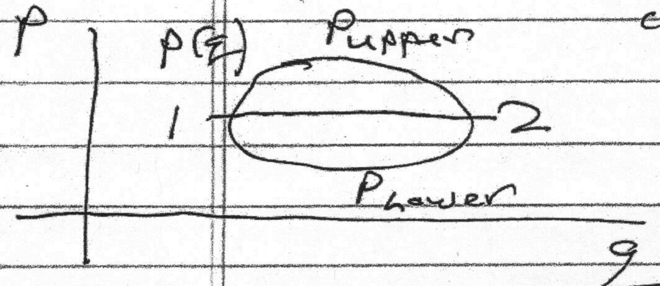
↓ phase volume conservation



(1D)

<u>Now</u>: $\int dp_i\, dq_i = \oint_C p_i\, dq_i$

circulation about C in phase space



$$A = \int_1^2 p_u(q)\, dq - \int_1^2 p_L(q)\, dq$$

enclosed area

$$= \int_1^2 p_u(q)\, dq + \int_2^1 p_L(q)\, dq$$

$$= \oint p\, dq$$

circulation

N.B.: Liouville Thm. analogous to <u>Kelvin</u> <u>circulation</u> <u>theorem</u> for isentropic fluids.

<u>Kelvin Thm.</u>

<u>i.e.</u> $\Gamma = \oint_C \underline{v} \cdot d\underline{l} = \text{const.}$

circulation

Euler Eqn

$$\frac{\partial u}{\partial t} + u \cdot \nabla u \;\overset{0}{\cancel{- \nu \nabla^2 u}} = -\frac{\nabla p}{\rho}$$

with viscosity → Navier-Stokes

incompressible: $\rho = \text{const}$

$$\frac{\nabla p}{\rho} = \nabla (p/\rho)$$

isentropic:

$$dE = T\,dS - p\,dV$$

$$\underset{\text{enthalpy}}{dW} = T\,dS + V\,dp$$

Enthalpy is Legendre Transform of Energy

isentropic $dS = 0$, $V = 1/\rho$

⇒

$$dW = \frac{dp}{\rho}$$

$$\frac{du}{dt} = -\nabla W \quad \text{(perfect gradient)}$$

$$\frac{d}{dt}\Gamma = \oint \frac{du}{dt} \cdot dl + \oint u \cdot \frac{dl}{dt}$$

$$= \oint -\nabla W \cdot dl + \int u \cdot du$$

$$= 0 + 0$$

$$\Rightarrow \boxed{\Gamma = \int \underline{v}\cdot d\underline{\ell} = \text{const} \quad \text{is Kelvin Thm.}}$$

Point: <u>Circulations</u> $\left\{\begin{array}{l}\oint \underline{p}\cdot d\underline{q} \\ \oint \underline{v}\cdot d\underline{\ell}\end{array}\right.$ are

dynamically conserved quantities in fluid flow (phase space or otherwise)

For proof of $\oint \underline{p}\cdot d\underline{q}$ conservation (arbitrary d), see Arnold

Now, Liouville Thm $\Rightarrow$ for any $A(q, p, t)$:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}A\right) + \frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}A\right) = 0$$

$$\left(\text{i.e.} \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{v}) = 0\right)$$

as $\quad \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = 0 \qquad$ $\left\{\begin{array}{l}\text{incompressible} \\ \text{phase space flow}\end{array}\right.$

$$= \frac{\partial A}{\partial t} + \dot{q}_i \frac{\partial A}{\partial q_i} + \dot{p}_i \frac{\partial A}{\partial p_i} = 0$$

and using HEOM:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \left(\frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i}\right)$$

$$\equiv \frac{\partial A}{\partial t} + \{A, H\}$$

$\{A, H\} \rightarrow \{A, B\} \equiv$ Poisson Bracket

$$\{A, B\} = \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}$$

→ P.B. is notational shorthand → evolution

→ P.B. defines operator relation, specifically a <u>non-commutative Lie Algebra</u>.

→ Bracket Properties

① $\{f, g\} = -\{g, f\}$ (anti-commutativity)

② $\{f+g, h\} = \{f, h\} + \{g, h\}$ (distributive)

③ $\{fg, h\} = f\{g, h\} + g\{f, h\}$

(associative – follows from derivative of product)

④ $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$

$= 0$

Jacobi Identity

→ see L&L for proof
(not instructive)
→ akin cross product rule.

Some key points:

a.) if $\{A, H\} = 0 \Rightarrow dA/dt = 0$

$\{$ with $A = A(q, p)$ indep. t. (i.e. $\partial A/\partial t = 0$) $\}$

$\Rightarrow A$ is COM.

b.) $\{A, H\} = 0$

$\{B, H\} = 0$

$\Rightarrow$ Jacobi identity:

$\{A, \{B, H\}\} + \{B, \{H, A\}\} + \{H, \{A, B\}\} = 0$

$\Rightarrow$

∞ $\{H, \{A, B\}\} = 0$

$\Rightarrow \{A, B\}$ is IOM.

ii) in particular: (Important!)

$\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$

$\{p_i, q_j\} = \delta_{i,j}$ $\Rightarrow [p_i, q_j] = -i\hbar\delta_{i,j}$ in QM

N.B.:

- Poisson bracket is notational short-hand

but

- also encapsulates key relationships of generalized coordinates.

(b) Canonical Transformations

May be useful to change variables. Need preserve Hamiltonian structure

$\Rightarrow$ in general, seek how transform:

$$\left.\begin{matrix} p_i \\ q_i \end{matrix}\right\} \rightarrow \left.\begin{matrix} P_i \\ Q_i \end{matrix}\right\}$$

"old" "new"

useful for:

$\rightarrow$ technical aspects of problem - i.e. change of variables

$\rightarrow$ writing in simplest form i.e. action-angle.

s/t Hamiltonian structure preserved

i.e. if $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$

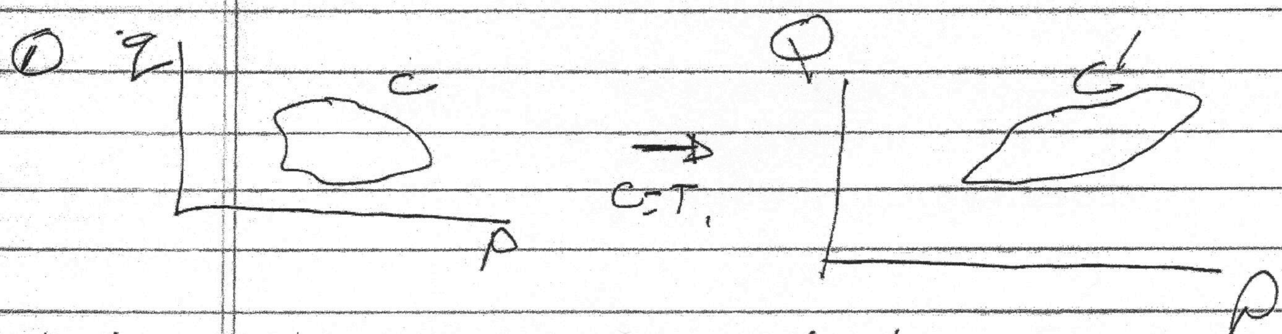
then $\dot{P}_i = -\frac{\partial H'}{\partial Q_i}$, $\dot{Q}_i = \frac{\partial H'}{\partial P_i}$

H' is new Hamiltonian.

Clearly such a transformation must:

① – preserve phase volume
② – preserve bracket relations

①



but enclosed areas must be equal:

$$\int_{A_C} dp_i\, dq_i = \int_{A_{C'}} dP_i\, dQ_i$$

so, must have:

$$\boxed{\frac{\partial(P_i, Q_i)}{\partial(p_i, q_i)} = 1}$$

Jacobian of transformation

N.B. Only require constant to re-scale variables. No loss generality to take as unity.

trivial example:

- how transform from q, p to $Q = q(t+\delta t)$, $P = p(t+\delta t)$?

i.e.

old	New
p	$P = p(t+\delta t)$
q	$Q = q(t+\delta t)$

new variables as time-advance

Obviously, use Hamiltonian EOMs

to $O(\delta t)$

$$P = p(t+\delta t) = p(t) + \delta t\,(dp/dt)$$

$$= p(t) - \delta t\,(\partial H/\partial q)$$

$$Q = q(t+\delta t) = q(t) + \delta t\,(dq/dt)$$

$$= q(t) + \delta t\,(\partial H/\partial p)$$

H generates transform. Is it canonical?

$$\frac{\partial(P, Q)}{\partial(p, q)} = \begin{vmatrix} \dfrac{\partial P}{\partial p}, & \dfrac{\partial P}{\partial q} \\[2ex] \dfrac{\partial Q}{\partial p}, & \dfrac{\partial Q}{\partial q} \end{vmatrix}$$

$$= \begin{vmatrix} 1 - \delta t \dfrac{\partial^2 H}{\partial p \partial q}, & -\delta t \left(\dfrac{\partial^2 H}{\partial q^2} \right) \\[2ex] \delta t \left(\dfrac{\partial^2 H}{\partial p^2} \right), & 1 + \delta t \dfrac{\partial^2 H}{\partial p \partial q} \end{vmatrix}$$

$$= 1 + \delta t \left(\frac{\partial^2 H}{\partial p \partial q} - \frac{\partial^2 H}{\partial p \partial q} \right)$$

$$+ \delta t^2 \left[\left(\frac{\partial^2 H}{\partial q^2} \right) \left(\frac{\partial^2 H}{\partial p^2} \right) - \left(\frac{\partial^2 H}{\partial p \partial q} \right)^2 \right]$$

$$= 1 + \delta t^2 \left\{ \text{Gaussian Curvature } H \right\}$$

so, after expansion to δt,

$$\frac{\partial(P,Q)}{\partial(p,q)} = 1 \quad , \quad \text{to } O(dt^2)$$

i.e. - have shown phase volume conservation to order of calculation

- transformation is canonical.

i.e. H 'generates' canonical transformation
$q(t),\ p(t) \rightarrow q(t+dt),\ p(t+dt)$

NB Can view H. E.O.Ms as sequence of canonical transformations
Simple example - see 12.

→ How to Transform Canonically? → 200B

- generally, seek transformation

$$\underset{\text{(old)}}{p,\ q} \rightarrow \underset{\text{(new)}}{P,\ Q}$$

where have:

- 2 independent variables + Gen. Fctn.
- 2 dependent variables

Simple Example : $\begin{cases} \text{Contact} \\ \text{Coordinate Change} \end{cases}$ Transformation

$$Q = Q(q)$$

What is canonical transformation ?

Point: Conserve phase volume ⇒

$$dP\, dQ = dp\, dq$$

$p, q \rightarrow$
$P, Q(q)$

$$dP\, dQ(q) = dp\, dq$$

$$dP \frac{dQ}{dq} dq = dp\, dq$$

$$dP = dp / dQ\cancel{q} = dp / dQ/dq$$

N.B. – choose $Q(q)$
– P forced by Liouville

Action - Angle Variables

→ Point:

- exact variables yielding EOM akin to adiabatic problem (approximate)

i.e. $\dot{I} \cong 0 \rightarrow \dot{I} = 0$

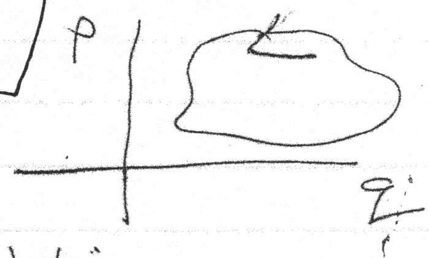
$$\dot{\theta} = \omega = \frac{\partial E}{\partial I} \rightarrow \omega = \frac{\partial H}{\partial I}$$

system integrable

- more generally, suggests
  - variables with action-as-momentum
  - $\dot{I} = -\frac{\partial H}{\partial \theta}, \quad \dot{\theta} = \frac{\partial H}{\partial I}$

Action–Angle Variables
(CL/L → canonical variables)

Key concept: $\oint p\,dq$



Phase Space Circulation

⟺ Poincare – Cartan Invariant

Canonical Transformations

⇒ specify transformation rules

so

Adiabatic Invariants

$\lambda = \lambda(t)$

$E = E(t)$

i.e. $\oint p\,dq \to$ COM with E, λ slow parametric variation

{ circulation as const motion

⇒

Action–Angle Variables

$E = $ const.

$\lambda = $ const.

closed system

{ circulation as variable (momentum)

⇒ integrability

{ phase space geometry, resonance

Action – Angle variables

⇒ seek variables (i.e. C.T.: $p, q \to I, \theta$) s/t: (P, Q)

$$H = H(I), \quad \text{so} \quad \dot{I} = 0$$

$$\dot{\theta} = \frac{\partial H}{\partial I} = \omega. \qquad \text{integrable}$$

$$\theta = \omega t + \theta_0$$

i.e. C.T. to conserved momentum, cyclic coordinate

⇒ C.T. is equivalent to integration of system.

$\boxed{\text{A/A are variables on which system integrated}}$

→ crudely : integrate via new variables s/t $I \to$ 'generalized radius'
$\theta \to$ " " angle

Transform details – skip.

so

$$p, q \to \theta, I$$

$$H(p, q) \to H'(I) \qquad \dot{I} = 0, \quad \dot{\theta} = \omega$$

C.T. : independent variables q, I (q, P)

$\Rightarrow$ Type II : $F_2 = F_2(q, P)$

so $p = \dfrac{\partial F_2}{\partial q}$, $Q = \dfrac{\partial F_2}{\partial P}$

$\Rightarrow$ $p = \dfrac{\partial F_2}{\partial q}$, $\theta = \dfrac{\partial F_2}{\partial I}$

but $p = \dfrac{\partial F_2}{\partial q}$ equiv. to $p = \dfrac{\partial S}{\partial q}$ from H-J theory

(always, for Type II)

so can write in terms action as generating function, i.e.

$$F_2(q, P) = F_2(q, I) = S(q, I).$$

so $\quad \theta = \dfrac{\partial S_0}{\partial I}, \quad p = \dfrac{\partial S_0}{\partial q}$

Now, Further:

$S_0 = S_0(q, I)$ indep. time; i.e. $\lambda = \lambda(t) = \text{const.}$

and $\quad H(q,p) \Rightarrow \underset{= E(I)}{H(I)},$ with θ cyclic

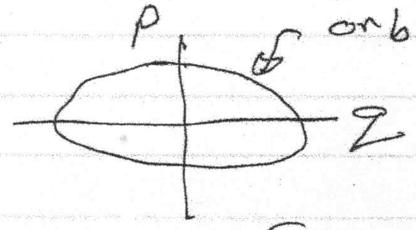
in new variables $\rightarrow$ EOM:

$$\Rightarrow \quad \dot{I} = -\frac{\partial H}{\partial \theta} = 0, \qquad \dot{\theta} = \frac{\partial H}{\partial I} = \underbrace{\omega(I)}_{\text{angular frequency}}$$

i.e. I and E constant.

Contrast: Adiabatic invariants $\Rightarrow$
$I \sim$ const, E evolves as ω evolves

Now,



$$I = \underbrace{\oint p \, dq}_{\text{1 circuit circulation}} \frac{1}{2\pi} = \underbrace{\int dp \, dq}_{\text{phase volume}}$$

another way:

$$S_0 = S_0(q, I) \quad , \quad p = \partial S_0 / \partial q$$

gen fctn. $\quad \theta = \partial S_0 / \partial I$

Action

so

$$\frac{d\theta}{dq} = \frac{\partial}{\partial q}\frac{\partial S}{\partial I} = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}$$

$$d\theta = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}\, dq$$

$\Rightarrow$

$$2\pi = \frac{\partial}{\partial I}\oint \frac{\partial S}{\partial q}\, dq$$

$$= \frac{\partial}{\partial I}\oint p\, dq.$$

$$\Rightarrow \quad \boxed{I = \oint \frac{p}{2\pi}\, dq \quad \rightarrow \text{Action Variable}}$$

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial I} = \frac{\partial E(I)}{\partial I} \equiv \omega(I)}$$

angle variable.

$I \rightarrow$ radius

$\omega \rightarrow$ winding rate, ~~frequency~~

Comparison / Contrast

Adiabatic Invariants	A-A Variables
$\lambda = \lambda(t)$, open loop	$\lambda = \lambda_0$ const, closed loop
$I = \oint_{E,\lambda} p\,dq \sim \left\{ \begin{array}{l} \text{approx} \\ \text{COM} \end{array} \right.$	$I = \oint p\,dq \quad \left\{ \begin{array}{l} \text{exact} \\ \text{COM} \end{array} \right.$
E varies with ω, $I \sim$ const.	E, I const. $\dot{I} = 0$ is H EOM
COM for multiple scale problems	Variables in which system is integrated i.e. $\dot{I} = 0$
1 adiabatic inv. per closed cycle (i.e. mirror) (separability implicit)	separable system $\Rightarrow$ 1 action variable/cycle.

so, $I = E/\omega$

$p = I \equiv$ "new" momentum

$H = E = I\omega$ so, $\dot{\theta} = \frac{\partial H}{\partial I} = \omega$

$\theta = \omega t + \theta_0$

$S = S(E, I) =$ [redacted] $\int_{q_0}^{q} dq \left(I\omega - \frac{\omega^2 q^2}{2}\right)^{1/2}$

2) For 2D,

$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_1^2 q_1^2}{2} + \frac{\omega_2^2 q_2^2}{2}$

$H = f(1) + f(2) = E$ separable!

$f(1) = \frac{p_1^2}{2} + \frac{\omega^2 q^2}{2} = E_1 \rightarrow$ const.

$f(2) = \frac{p_2^2}{2} + \frac{\omega^2 q^2}{2} = E_2 \rightarrow$ const.

[redacted]

so, for action variables I_1, I_2:

$$I_1 = \frac{1}{2\pi}\oint p_1\, dq = \frac{1}{2\pi}\oint p_1(q_1)\, dq_1 = \frac{E_1}{\omega_1}$$

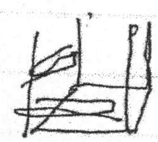
$$I_2 = E_2/\omega_2$$

$$H(I_1, I_2) = E = E_1 + E_2$$
$$= I_1\omega_1 + I_2\omega_2$$

⇝ separable, so
⇝ additive form of H in A-A variables

3) Free Particle in 2D $\begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$ (hard wall)

$$H = \frac{1}{2m}(p_x^2 + p_y^2)$$



→ 2 Degs freedom ⇒ 2 I's, 2 θ's

$\therefore$ $$I_1 = \frac{1}{2\pi}\oint p_x\, dx$$

$$I_2 = \frac{1}{2\pi}\oint p_y\, dy$$

$$\oint p_x \, dx = \int_0^a p_{x_+} \, dx + \int_a^0 p_{x_-} \, dx$$

$p_{x_+} = -p_{x_-}$ (reverse when bounce off wall)

$$\oint p_x \, dx = 2a \, |p_x|$$

$$\therefore \quad I_1 = \frac{a}{\pi} |p_x|$$

$$I_2 = \frac{b}{\pi} |p_y|$$

So
$$H = E = \frac{p_x^2 + p_y^2}{2m}$$

$$= \frac{\pi^2}{2m} \left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2} \right)$$

$$\omega(I_{1,2}) = \frac{\partial E(I_1, I_2)}{\partial I_{1,2}} = \frac{\pi^2}{m} \frac{I_1}{a^2}, \; \frac{\pi^2}{m} \frac{I_2}{b^2}$$

2 Points:

(a) contrast:

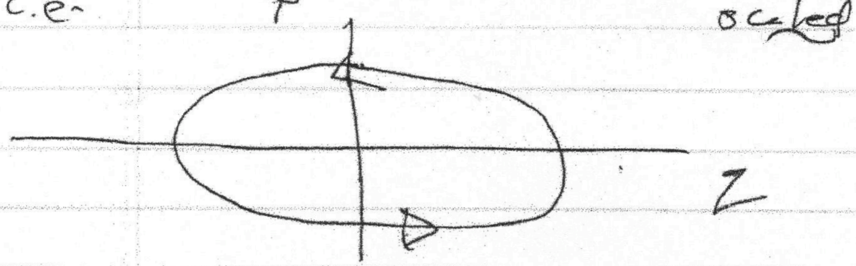
~> H.O. $\qquad \omega(I) = \omega_0 = \text{const.}$

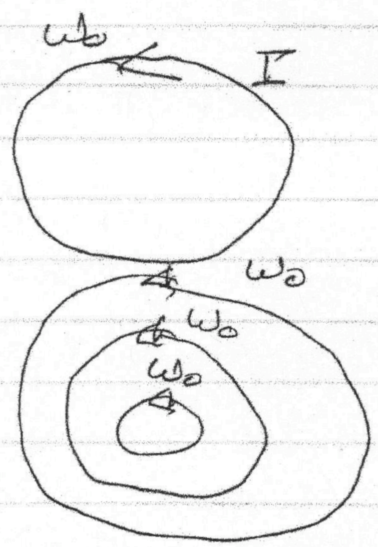
$$\partial\omega / \partial I = 0$$

$I\omega_0 = E$ → constant frequency

→ no shear in winding rate

i.e.





i.e.

[and all I contours have same rotation frequency ω_0

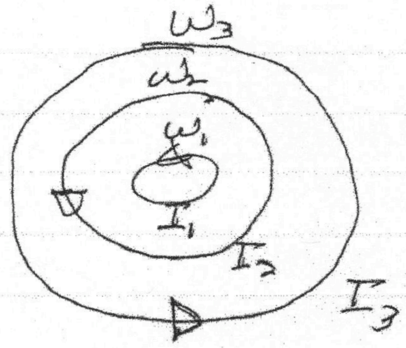
→ box $\quad \omega(I) = \frac{\pi^2}{ma^2} I$

$\partial\omega(I)/\partial I \neq \qquad \omega_i \sim |p|$

⇒ winding rate varies with I

⇒ "shear"

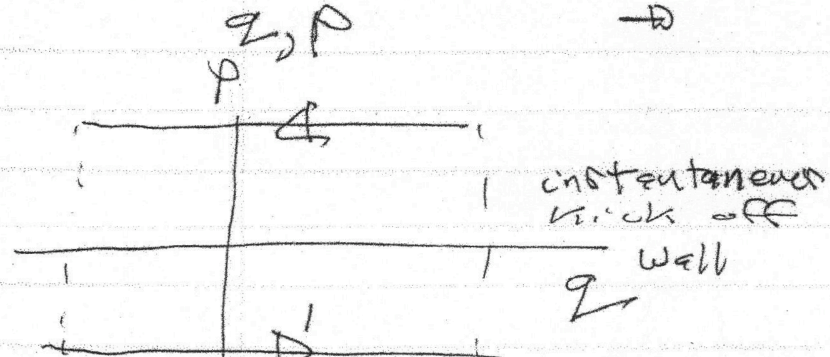
i.e.

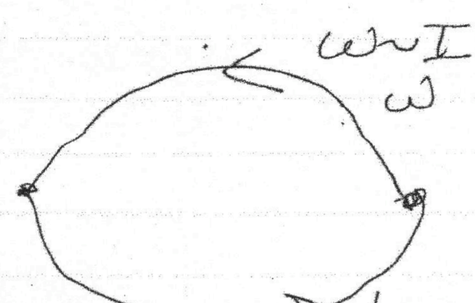


winding rate increases with I (i.e. $\omega \sim I$)

⇒ differential rotation

$q, p \quad \rightarrow \quad I, \omega$





i.e. top box – top circle, etc.

[redacted]

? H.O. is $\underline{\text{linear}}$ problem, with $\partial\omega/\partial I = 0$

Box has $\partial\omega/\partial I \neq 0$, yet is linear too ?!

Why ?

n.b. Consider general 1D potential:

$$H = p^2 + V(q)$$

$$I = \oint \frac{p\,dq}{2\pi} = \frac{1}{2\pi}\oint \left[E - V(q)\right]^{1/2} dq$$

$$= \underline{I(E)}$$

$$\omega = \partial E(I)/\partial I$$

now, for $V(q) \sim e\,q^4$

$$I \sim c' E^{3/4}$$

$\Rightarrow \; E \sim c\,I^{4/3}$ so $\omega(I) \sim c'' I^{1/3}$

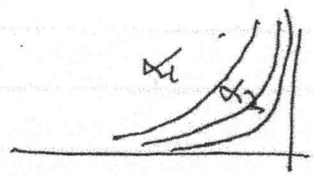
shear!

$\Rightarrow$ Nonlinearity develops from $V \sim \xi^{\alpha}$ potential for $\alpha > 2$.

$\therefore$ View hard wall as a limiting case

i.e.

$$V = V_0 \left(x/a\right)^{\alpha}$$



$\alpha \gg 1$

so hard wall boundary condition appears as nonlinearity due high high powers implicit in piecewise continuous potential.

② Reln. QM

classically: $$H = E = \frac{\pi^2}{2m}\left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2}\right)$$

if $\left.\begin{array}{l} I_1 \to n\hbar \\ I_2 \to m\hbar \end{array}\right\}$ quantize action variables

$$E = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \to$$ eigenstates of free QM particle in box

[redacted]

aside: Can observe correspondance

<u>Classical</u> | <u>Quantum</u>

$I = E/\omega$ | $E = (N + 1/2)\hbar\omega$

$H = I\omega$

\# quanta (quantum #) ⟷ occupation

$\boxed{\text{suggests view } I \text{ as classical \# of excitations/waves} \Rightarrow \text{exciton density}}$

straightforward to generalize: (linear wave)

$I = E/\omega \quad \Rightarrow \quad N(\underline{k}, \underline{x}) = \underline{E}(\underline{k}, \underline{x})/\omega_k$

I: linear H.O.

N: Action Density as Wave Density, # waves

E: wave energy density

ω_k: wave frequency

→ General Properties of Motion in s dimensions.

Now, consider:
- s degrees of freedom (arbitrary system)
- separable H.-J. equation

$$S = \sum_{i=1}^{s} S_i(\mathcal{E}) \qquad \text{(i.e. integrable)}$$

∴ can define s action variables I_i

$$I_i = \oint \frac{p_i\,dq_i}{(2\pi)} \qquad \text{i.e. } s\text{-IOMs.}$$

and $\theta_i = \partial S_0/\partial I_i$ angle variables

so

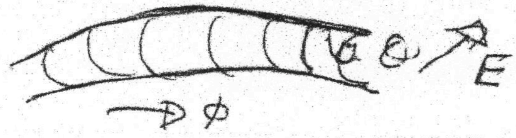
$$\dot{I}_i = 0$$

$$\dot{\theta}_i = \omega_i(E)\,t + t_0$$
$$\omega_i(E) = \partial E/\partial I_i$$

i.e. for $s = 2$

$$\dot{I}_1 = \dot{I}_2 = 0 \qquad \omega_{1,2} = \partial E/\partial I_{1,2}$$

$$\theta_{1,2} = \omega_{1,2}(E)\,t + t_0$$



change I
→ change surf
→ nested surfaces.

∴ phase space is 2 torus. Fixed E ⇒ motion on toroidal surface.

[In general, phase space is s-torus.]

$\theta = \omega_1(E)\,t$

$\phi = \omega_2(E)\,t$

$\theta = \dfrac{\omega_1(E)}{\omega_2(E)}\,\phi$

→ now, for any $F(\underline{q}, \underline{p})$, can write:

fourier series

$$F = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_s} A_{l_1, l_2 \ldots s} \exp\left[i(l_1\theta_1 + l_2\theta_2 + \cdots l_s\theta_s)\right]$$

$l_1, l_2, \ldots l_s$ integers. ⇒ define vector $\underline{l}$

equivalently:

ω_i § $\underline{l}\cdot\underline{\omega}\,t$

$$F = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_s} A_{l_1, l_2 \ldots l_s} \exp\left[it\left(\underline{l} \cdot \frac{\partial E}{\partial \underline{I}}\right)\right]$$

$$\underline{l} \cdot \frac{\partial E}{\partial I} = l_1 \frac{\partial E}{\partial I_1} + l_2 \frac{\partial E}{\partial I_2} + \cdots l_s \frac{\partial E}{\partial I_s}.$$

Now, in general:

→ frequencies not commensurate, so F not periodic i.e. $\underline{l} \cdot \partial E / \partial \underline{I}$ irrational

→ indeed, system generally not periodic in any coordinate (except for special E).

But, for sufficient time, system will come arbitrarily close, to starting point. → [Poincaré Recurrence Thm !]

sufficient windings. i.e. {trajectory ergodically covers surface of torus}

∴ motion is "conditionally" periodic.

But; degeneracy happens!

- degeneracy: $n\omega_i = m\omega_j$
- all ≤ commensurate ⇒ complete degeneracy

So, as in Kepler problem, ⇒ degeneracy implies reduction in number of independent I_i. Why?

Commensurate frequencies $\Rightarrow$

$$n_1 \omega_1 = n_2 \omega_2$$

$$n_1 \frac{\partial E}{\partial I_1} = n_2 \frac{\partial E}{\partial I_2}$$

so $E = E(n_2 I_1 + n_1 I_2)$

i.e. - energy depends on sum of action variables
linear superposition

$\Rightarrow$

- degeneracy.

$\Rightarrow$

- can make canonical transformation so $E = E(I')$, only.

$\Rightarrow$

$\therefore$ in degenerate motion, there is an increase in the number of one-valued integrals of the motion, relative to non-degenerate case.

i.e. non-degenerate motion - s degs freedom

$2s-1 \rightarrow$ IOM'S

$\left\{ \begin{array}{l} s \text{ values } I_i \Rightarrow \text{ single valued } I_i \\ s-1 \quad \text{values a/a/ } \theta_i \, \partial E/\partial I_k - \theta_k \, \partial E/\partial I_i \end{array} \right.$

note: $S-1$ values → phases (i.c.'s) of angle variables.

→ **not** single valued.

but if degeneracy, note though:

→ $n_1\theta_1 - n_2\theta_2$ not single valued

it **is**, to addition of 2π!

so

→ $\sin(n_1\theta_1 - n_2\theta_2)$ **is** single valued.
(etc)